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Notes for a presentation; tentative English title:
" '$\mathbf{Set}^{\mathbf{C}}$ is a topos' has a syntactical proof."

This is a very preliminary version.
It doesn't even have an abstract yet :-)
Please see my home page at `http://angg.twu.net/`
or its mirror at `http://www.mat.puc-rio.br/~edrx/`
for more math stuff and for ways to contact me.

Table of Contents

Introduction (for non-toposophers)

I have the impression that a significant part of mathematics is "syntactic"; "syntactical proofs" are much simpler than proofs in the usual form — in fact they are often trivial.

If we could develop the tools to do more mathematics syntactically we would make several important theorems almost trivial, and people would be able to learn them much faster, and to reconstruct their proofs from their statements very quickly.

But what is a syntactic proof?

Let's first understand what people mean with "a category $\mathbf{A}$ is a topos and so we can interpret first-order logic in it". We will get down to the details of what is a topos later — this is just motivation.

Set is a topos, and the tree below shows very roughly how to interpret a sample formula — $\forall x.P(x) \vee (\exists y.Q(x,y,a))$ — in it. Through a series of "elementary" categorical steps we build an arrow $a \to \omega_{\forall P(x) \vee (\exists y.Q(x,y,a))}$ from $x \to \omega_{P(x)}$ and $x, y, a \to \omega_{Q(x,y,a)}$.

$$
\begin{array}{c}
\begin{array}{c}
\dfrac{x \overset{P}{\to} \omega[P(x)]}{x, a \to \omega[P(x)]}
\quad
\dfrac{\dfrac{x, y, a \overset{Q}{\to} \omega[Q(x,y,a)]}{x, a \to (y \to \omega[Q(x,y,a)])} \quad \begin{array}{c} \\ \hline\hline (y \to \omega) \overset{\exists y}{\to} \omega \end{array}}{x, a \to \omega[\exists y.Q(x,y,a)]}
\\ \hline\hline
x, a \to \omega[P(x) \vee (\exists y.Q(x,y,a))]
\\ \hline
a \to (x \to \omega[P(x) \vee (\exists y.Q(x,y,a))])
\\ \hline\hline
a \to \omega[\forall P(x) \vee (\exists y.Q(x,y,a))]
\end{array}
\end{array}
$$

$\mathbf{E}_x, \mathbf{E}_y, \ldots$ are sets, and objects of **Set**
$\mathbf{E}_{x,y} := \mathbf{E}_x \times \mathbf{E}_y$
$\mathbf{E}_{x \to y} := \{ f \mid f : \mathbf{E}_x \to \mathbf{E}_y \}$
$\mathbf{E}_\omega := \{ \dashv, \top \}$
$\mathbf{E}_\top := \{ \top \}$
$\mathbf{E}_{x|_{P(x)}} := \{ x \in \mathbf{E}_x \mid P(x) \}$
$\mathbf{E}_{\top|_P}$ is a singleton if P is true, and an empty set if P is false.
There is a natural inclusion $\top \hookrightarrow \omega$.
$P(x) \to Q(x)$ (for all x) iff there is a natural inclusion $x|_{P(x)} \hookrightarrow x|_{Q(x)}$.
There is a natural construction for $a, b \to b, a$, and it corresponds to a proof of $P \wedge Q \to Q \wedge P$:

$$
\dfrac{\dfrac{}{a, b \to b} \quad \dfrac{}{a, b \to a}}{a, b \to b, a}
\qquad
\dfrac{\dfrac{}{\pi'_{A,B}} \quad \dfrac{}{\pi_{A,B}}}{\langle \pi'_{A,B}, \pi_{A,B} \rangle}
\qquad
\dfrac{\dfrac{}{P \wedge Q \to Q} \quad \dfrac{}{P \wedge Q \to P}}{P \wedge Q \to Q \wedge P}
$$

$$
\dfrac{\dfrac{\dfrac{[a,b]^1}{b} \quad \dfrac{[a,b]^1}{a}}{b, a}}{a, b \to b, a}\,1
\qquad
\dfrac{\dfrac{\dfrac{[p]^1}{\pi'(p)} \quad \dfrac{[p]^1}{\pi(p)}}{(\pi'(p), \pi(p))}}{\lambda p.(\pi'(p), \pi(p))}\,1
\qquad
\dfrac{\dfrac{\dfrac{[P \wedge Q]^1}{Q} \quad \dfrac{[P \wedge Q]^1}{P}}{Q \wedge P}}{P \wedge Q \to Q \wedge P}\,1
$$

All the natural constructions for an arrow that would deserve the name $a, b \to b, a$ give the same result, so we can use the "type" of this arrow as its name.

The case of $a, b \to b, a$ is too simple to be really interesting, but in many cases an important property can be stated as the existence of a natural arrow of a certain type, $\alpha \to \beta$, say, plus checking that this arrow obeys certain conditions — and these conditions are usually like checking that some equations are satisfied, which is generally very easy. So, once the problem is stated, we may proceed as follows: 1) we try to find a natural construction that gives an arrow of the desired type; 2) we check that it obeys the equations. In many cases steps (1) and (2) are almost trivial; in these cases we say that the problem is "syntactical", and we consider that the only non-trivial step was stating the "names" of the things involved.

I would like to be able to show that some concepts and theorems of Calculus and Differential Geometry are syntactic, but there's much work to do before being able to formalize that; and this work would involve infinitesimals, and thus (either via Non-standard Analysis or via its psychedelic counterpart, Synthetic Differential Geometry), Category Theory.

But what is a topos?

A topos is a category with binary products (like $\mathbf{E}_{x,y}$), a terminal object (like $\mathbf{E}_\top$), pullbacks (like $\mathbf{E}_{(a,b)|_c}$; the arrows $a \to c$ and $b \to c$ are implicit), exponentials (like $\mathbf{E}_{x \to y}$) and a classifier object $\mathbf{E}_\omega$, with a "true" arrow $\top \to \omega$. Everything else is defined from this; for example [note: this is a version of the definitions in Bell's book, and L8 and L9 may be slightly wrong...],

$$
\begin{array}{lll}
(L1) & \omega[P \leftrightarrow Q] & := & \omega[\omega[P] = \omega[Q]] \\
(L2) & \omega[\top] & = & \omega[\omega' = \omega'] \\
(L3) & \omega[P \wedge Q] & := & \omega[(\omega_P, \omega_Q) = (\omega_\top, \omega_\top)] \\
(L4) & \omega[P \to Q] & := & \omega[\omega_P = \omega[P \wedge Q]] \\
(L5) & \omega[\forall b.P] & := & \omega[(b \to \omega_P) = (b \to \top \to \omega)] \\
(L6) & \omega[\bot] & := & \omega[\forall \omega[P].P] \\
(L7) & \omega[\neg P] & := & \omega[P \to \bot] \\
(L8) & \omega[P \vee Q] & := & \omega[\forall \omega[R].(((P \to R) \wedge (Q \to R)) \to R)] \\
(L9) & \omega[\exists b.P] & := & \omega[\forall \omega[Q].((\forall b.(P \to Q)) \to Q)] \ \ (?)
\end{array}
$$

$\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ an "index set", is a topos.

$\mathbf{Set}^{\bullet \to \bullet}$ is a topos, and its logic is not boolean: in it a $\top|_P$ can be different from the corresponding $\top|_{\neg\neg P}$.

$\mathbf{Set}^{\mathbf{C}}$, for a (small) category $\mathbf{C}$, is a topos — this generalizes all the above cases.

We are going to see that "$\mathbf{Set}^{\mathbf{C}}$ is a topos" is syntactical; and in the way we will learn basic category theory, both in the "types" notation an ind the usual notation, and see how to translate between them.

The notation of typical points

Our "notation of typical points" is based on the fact that when we have a set X we usually choose names like x, x_0, x_1, x', etc., to denote its points; this idea will be taken to an extreme: X will be called the "space of 'x's", and we'll write $\mathbf{E}_x = X$ for this; a pair made of an a and a b is written as (a,b), and so $\mathbf{E}_{(a,b)} = A \times B$; a function that takes each 'x's to a 'y' is an '$x \to y$', and so $\mathbf{E}_{x \to y} = Y^X$. Many other notations — and many abuses of the notation we already presented, especially of the case '$x \to y$' — will appear as the text goes on.

Categories

We will frequently want to treat $\mathbf{E}_a$ not as as set, but as an object of a category — it will be written as $\mathbf{O}_a$ then. Certain categories will have objects that deserve to have names like, say, $\mathbf{O}_{a \to b}$; in a case like this the notation $a \to b$ will be ambiguous, and "an '$a \to b$' " can be either a "point" of $\mathbf{O}_{a \to b}$, if this notion makes sense, or a morphism from $\mathbf{O}_a$ to $\mathbf{O}_b$, or a point of the "space of morphisms from $a \to b$", $\mathbf{E}_{a \to b}$ (an hom-set); we will see that allowing such ambiguities will be much more convenient than confusing.

Some arrows in our categories will be written without "decorations" — like $a \to b$ instead of $a \xrightarrow{f} b$. This will indicate either that the given arrow is "natural enough" (in a sense that will be made precise later) to deserve to have the empty decoration as his name, or that we are speaking of a generic arrow of "type" $a \to b$. We will be helped by the context and by our use of articles: "$x \to y$" has a quite different meaning from "a $x \to y$".

Arrows without decorations will be called "unnamed arrows" for brevity.

If we know an $a \to b$ and a $b \to c$ then we have a "natural choice" for the unnamed arrow $a \to c$: the composite $a \to b \to c$. The general rule is that we will always use the most "natural" choice for the unnamed arrows (an for other unnamed objects), and the staggering truth is that we will need named objects only very rarely.

Sometimes it will be convenient (for drawing diagrams, say) to have several names to the same object; if we see an identity arrow between two objects with slightly different names then it will be obvious that they are the same.

Some notation and language: $\mathbf{Cat}_a$ is the category whose "typical object" is $\mathbf{O}_a$; $\mathbf{A} = \mathbf{Cat}_a$, $\mathbf{B} = \mathbf{Cat}_b$, and so on, but it will be common to have $\mathbf{O}_a$ and $\mathbf{O}_b$ as objects of $\mathbf{C}$ and $\mathbf{O}_x$ as an object of any category whose name is a single letter; other abuses are permitted but they will be less frequent.

Another way to say that $\mathbf{O}_a$ is an object of $\mathbf{A}$ is "$\mathbf{O}_a$ is an $\mathbf{o}_\mathbf{A}$"; similarly, a morphism $a \to b$ of $\mathbf{A}$ is a $\mathbf{m}_\mathbf{A}$.

Special arrows in a category

If an arrow $a \to b$ has an "inverse" $b \to a$ — i.e., an arrow such that $a \to b \to a$ and $b \to a \to b$ are identities — then $a \to b$ is an *isomorphism*, and it may be written as $a \leftrightarrow b$.

If an arrow $b \to c$ is a monic, i.e., if the operation $\mathbf{E}_{a \to b} \to \mathbf{E}a \to c$ induced by composition is injective, then we may write $b \to c$ as $b \hookrightarrow c$. The tradition is to use '$\rightarrowtail$' for monics and '$\hookrightarrow$' for injectives, but we will use only '$\hookrightarrow$' (to stress the fact that all our syntax is inspired in **Set**, where the two concepts are equivalent).

The concept dual to monic is "epi" (that corresponds to surjection in **Set**), and if $a \to b$ is epi we may write it as $a \twoheadrightarrow b$.

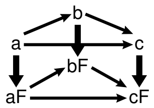
Duals

$\mathbf{Cat}_{a^{\mathrm{op}}} = \mathbf{A}^{\mathrm{op}}$ is $\mathbf{A}$ with the arrows reversed; an $\mathbf{O}_{a^{\mathrm{op}}}$ is just $\mathbf{O}_a$, but $a^{\mathrm{op}} \to b^{\mathrm{op}}$ is $b \to a$ (or $a \leftarrow b$) in disguise.

Functors

"An a" is a point of $\mathbf{E}_a$; "an $a^{[\,]}$" is finite list of 'a's (example: if $\mathbf{E}_a = \{a_1, a_2, a_3\}$ then $[a_3, a_2, a_2]$ and $[\,]$ are '$a^{[\,]}$'s). Each $\mathbf{E}_{a^{[\,]}}$ can be considered as a monoid — a set of lists plus an operation of composition between these lists that is associative and has a neutral element. Each $a \to b$ induces a map $a^{[\,]} \to b^{[\,]}$ that is an homomorphism of monoids; so what we're seeing is a functor $\mathbf{Set} \Rightarrow \mathbf{Mon}$, that we write as $a \Rightarrow a^{[\,]}$, as it is convenient do denote a functor like this one just by its action on objects, using a wide arrow: this functor takes $\mathbf{O}_a \in \mathbf{Set}$ to $\mathbf{O}_{a^{[\,]}} \in \mathbf{Mon}$, $\mathbf{O}_b$ to $\mathbf{O}_{b^{[\,]}}$, etc. In this case we have a good way to name the image of an arbitrary arrow by the functor (f goes to $f^{[\,]}$), but it will be very common to know only that a functor takes unnamed arrows to unnamed arrows ($a \to b$ to $a^{[\,]} \to b^{[\,]}$, for example).

Functors respect composition and identity arrows; in the diagram below once we choose the values of $a \to b$ and $b \to c$ all natural ways to build an arrow that would be named $a^F \to c^F$ give the same result.



Contravariant functors are like common ("covariant") functors but they reverse the arrrows; when we can't decide if it is better to write, say, $a \Rightarrow a^{\mathrm{op}}$ or $a^{\mathrm{op}} \Rightarrow a$ we are allowed to write just $a \Rightarrow^{\mathrm{op}} a$.

Bicategories

An object $\mathbf{O}_{(a;b)}$ of $\mathbf{A} \times \mathbf{B}$ is a pair of objects $(\mathbf{O}_a, \mathbf{O}_b)$; a morphism $(a;b) \to (a';b')$ is a pair $(a \to a'; b \to b')$.

There are obvious projection functors $(a;b) \Rightarrow a$ and $(a;b) \Rightarrow b$.

Hom

Let $\mathbf{C}$ be a small category. Better yet, let's not mind which categories are small and which are not, and whether the sets we are considering are really sets or just classes; let's leave these distinctions for when the basic ideas were made clear.

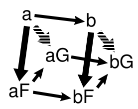
A good name for the famous functor $\mathrm{Hom} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \Rightarrow \mathbf{Set}$ is $(a^{\mathrm{op}}; b) \Rightarrow (a \to b)$, as it takes each pair $(\mathbf{O}_a, \mathbf{O}_b)$ of objects of $\mathbf{C}$ to the set $\mathbf{E}_{a \to b}$ (an object of **Set**) of morphisms $a \to b$, and it is contravariant in the first "$\mathbf{C}$"; its action on objects and morphisms can be described by the trees below:

$$\frac{\mathbf{o}_{\mathbf{C}^{\mathrm{op}} \times \mathbf{C}}}{\mathbf{o}_{\mathbf{Set}}} \qquad \frac{\mathbf{O}_{a^{\mathrm{op}}}; \mathbf{O}_b}{\mathbf{E}_{a \to b}}$$

$$\frac{\mathbf{m}_{\mathbf{C}^{\mathrm{op}}\times\mathbf{C}}}{\mathbf{m}_{\mathbf{Set}}} \qquad \frac{(a^{\mathrm{op}};b) \to (a'^{\mathrm{op}};b')}{(a \to b) \to (a' \to b')} \qquad \frac{(a \leftarrow a'; b \to b')}{(a \to b) \to (a' \to b')}$$

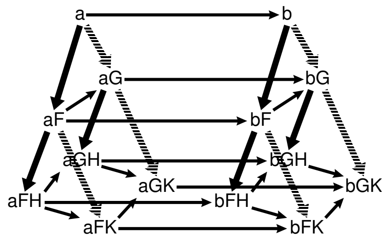
Natural transformations

A natural transformation between two functors $F, G : \mathbf{A} \to \mathbf{B}$ (i.e., from $a \Rightarrow a^F$ to $a \Rightarrow a^G$) is written as $a \overset{\bullet}{\to} (a^F \to a^G)$, as it takes an object $\mathbf{O}_a$ to a morphism $a^F \to a^G$ (if $T = a \overset{\bullet}{\to} (a^F \to a^G)$ then $T(\mathbf{O}_a)$ is $a^F \to a^G$). We require that the square below commutes, and so for each choice of $a \to b$ the two natural ways to build an $a^F \to b^G$ give the same result.



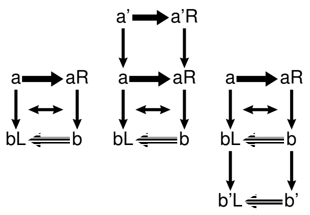
If all arrows returned by a natural transformation are isomorphisms, monics or epis we may indicate this in the name of the natural transformation: $a \overset{\bullet}{\to} (a^F \leftrightarrow a^G)$, $a \overset{\bullet}{\to} (a^F \hookrightarrow a^G)$, $a \overset{\bullet}{\to} (a^F \twoheadrightarrow a^G)$.

There are several ways to compose functors and natural transformations, and they generally commute beautifully; the unbelieving reader should check that in the diagram below everything commutes and all ways of obtaining a natural transformation $a \overset{\bullet}{\to} (a^{FH} \to a^{GK})$ are equivalent.



Adjunctions

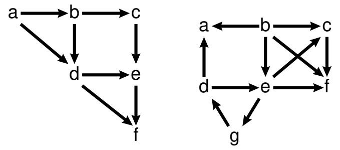
An adjunction between two functors , $a \Rightarrow a^R : \mathbf{A} \Rightarrow \mathbf{B}$ and $b \Rightarrow b^L : \mathbf{B} \Rightarrow \mathbf{A}$ is an operation that given two objects $\mathbf{O}_a$ and $\mathbf{O}_b$ (of $\mathbf{A}$ and $\mathbf{B}$, respectively) returns a bijection of hom-sets $(a \to b^L) \leftrightarrow (a^R \to b)$ such that the two bigger diagrams below always commute:



See the appendix 1 for the exact definition of commutativity that we are using; the base objects in the middle diagrams are $a' \to a$ and $a \to b^R$ (and thus also $\mathbf{O}_{a'}$, $\mathbf{O}_a$ and $\mathbf{O}_b$) and on the right diagram, $a^R \to b$ and b' (plus $\mathbf{O}_a$, $\mathbf{O}_b$, $\mathbf{O}_{b'}$); note that the bijections $(a' \to b^L) \leftrightarrow ({a'}^R \to b)$ must be taken in consideration when we speak of the commutativity of those diagrams.

The conditions on the bijections $(a \to b^L) \leftrightarrow (a^R \to b)$ are imposed to make certain diagrams commute, given that certain subdiagrams of them commute; the conditions on the definitions of categories, functors and natural transformations can be thought in exactly the same way.

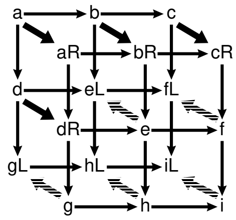
The fact that the composition of morphisms is associative makes every planar diagram with exactly one "source" and one "drain" in each cell (e.g., the one on the left below) commute if each of its cells commute; the diagram at right below shows several things that can happen in a "bad" diagram that does not obey these conditions.



Now take any commuting diagram in a category $\mathbf{C}$, and any functor $\mathbf{C} \Rightarrow \mathbf{D}$; form a new diagram with the old one plus its image in the category $\mathbf{D}$, but take care not to identify the images of objects and morphisms that were different in $\mathbf{C}$; even if, say, $F(A) = F(B)$, we must give different names for $F(A)$ and $F(B)$ in our diagram, and pretend that they are not especially related at all... the new diagram, made of the old one plus its image, will also commute; and of course we don't need to have $\mathbf{C} \neq \mathbf{D}$, if we are careful to avoid identifications.

With a natural transformation between two functors from $\mathbf{C}$ to $\mathbf{D}$ something similar happens, but the original commuting diagram in $\mathbf{C}$ gets two disjoint copies in $\mathbf{D}$, plus one extra arrow in $\mathbf{D}$ for each object of $\mathbf{C}$; the big diagram will also commute.

The correspondent commutativity rule for diagrams involving adjunctions is too boring to state, so we will only give an example.



Review of the typical point notation

$\mathbf{O}_a$ is the "object of 'a's", in a certain category; if this category is **Set** then this $\mathbf{O}_a$ can also be written as $\mathbf{E}_a$, the "space of 'a's".

An arrow $a \to b$ is a morphism from $\mathbf{O}_a$ to $\mathbf{O}_b$; sometimes an unspecified morphism, sometimes a specific morphism, the "unnamed morphism" from $\mathbf{O}_a$ to $\mathbf{O}_b$; we have a function that converts names of morphisms to morphisms; $a \stackrel{f}{\to} b$, for example, another name for a specific morphism from $\mathbf{O}_a \to \mathbf{O}_b$. The '$f$' is the "decoration" of the (arrow of the) morphism, and "unnamed" means "with an empty decoration".

We have chosen this function that interpret names in a way that it behaves as naturally as possible, in an informal sense, that is even allowed to change with the context; for example, we will often impose that the result of composing two unnamed morphisms $a \to b$ and $b \to c$ is the unnamed morphism $a \to c$.

$\mathbf{Cat}_a$ is the category where the object $\mathbf{O}_a$ lives.

$a \Rightarrow a^F$ is a functor from $\mathbf{Cat}_a \to \mathbf{Cat}_{a^F}$; when this is interpreted as a specific functor (i.e., most of the time) the notation implies that it takes each object $\mathbf{O}_a$ and takes it to $\mathbf{O}_{a^F}$ – i.e., it takes the name of an object and adds a decoration 'F' to it.

It is so common that there is a "natural" way to choose the images of morphisms once we are told how a functor acts on (names of) objects that we will generally denote a functor only by its action on objects; and these functors will take unnamed morphisms to unnamed morphisms.

An arrow with a curly tail, like $a \hookrightarrow b$, denotes a monic, or an injection. An arrow like '$a \leftrightarrow b$' denotes an isomorphism or bijection. Having an identity arrow between two objects with different names, like $a' \stackrel{\mathrm{id}}{\to} a''$, implies that we are using two different names for the same object (generally to let us write more arrows as unnamed arrows).

A natural transformation between two unnamed functors $a \Rightarrow a^F$ and $a \Rightarrow a^G$ (unnamed in the sense that their arrows have no decorations) is denoted by $a \stackrel{\bullet}{\to} (a^F \to a^G)$.

We write (a, b) for the product of $\mathbf{O}_a$ and $\mathbf{O}_b$, as in **Set** an (a, b) (i.e., an element of $\mathbf{E}_{(a,b)}$) can be seen as a pair made of an a and a b.

We use ';'s for more formal pairs; an $\mathbf{O}_{(a;b)}$ is a pair $(\mathbf{O}_a, \mathbf{O}_b)$, an object of the bicategory $\mathbf{Cat}_a \times \mathbf{Cat}_b$ (also called $\mathbf{A} \times \mathbf{B}$ or $\mathbf{Cat}_{(a;b)}$).

The projection functors $(a; b) \Rightarrow a$ and $(a; b) \Rightarrow b$ are typical cases of a functor where we want the target object to have a simpler name than the original object; likewise with some care we can allow functors like $a^F \Rightarrow a^G$, that syntactically work by removing an F from the name of an object and placing a G in its place, without even requiring that we have a meaning for $\mathbf{O}_a$ and $\mathbf{Cat}_a$.

Adjunctions are written as square diagrams (like the ones below) showing the bijection between hom-sets. The two most common examples are $(s \to m^U) \leftrightarrow (s^{[\ ]} \to m)$, between the "free monoid" and "underlying set" functors between **Set** and **Mon**, and $(a \to (b \to c)) \leftrightarrow (a, b \to c)$ between the product and the exponential by $\mathbf{O}_b$ in a cartesial closed category (e.g. **Set**).

$$\begin{array}{ccccccc}
s^{[\,]} & \Leftarrow & s & \qquad & a,b & \Leftarrow & a \\
\downarrow & \leftrightarrow & \downarrow & & \downarrow & \leftrightarrow & \downarrow \\
m & \Rightarrow & m^U & & c & \Rightarrow & b \to c
\end{array}$$

We also have a "less precise" notation for categories, and some extra symbols to help converting between our notations and the usual ones. *This part of the notation has many rough edges and won't be used much, but, anyway:* instead of saying that $\mathbf{O}_a$ is an object of $\mathbf{A}$ and that $a \to b$ is a morphism of $\mathbf{A}$, we can say that $\mathbf{O}_a$ is an $\mathbf{o_A}$ and that $a \to b$ is a $\mathbf{m_A}$; an element of a space $\mathbf{E}_a = A$ is an $\mathbf{e}_A$; a functor F from $\mathbf{A}$ to $\mathbf{B}$ is a functor $F : \mathbf{A} \Rightarrow \mathbf{B}$, or just "a functor $\mathbf{A} \Rightarrow \mathbf{B}$"; $\mathbf{A} \Rightarrow \mathbf{B}$ is also used as the name of the category that has these functors as its objects, and natural transformations between them as its morphisms; $\mathbf{A} \times \mathbf{B}$ is the bicategory that is the product of $\mathbf{A}$ and $\mathbf{B}$.

We will sometimes use $\mathbf{O}[a]$, $\mathbf{E}[a]$, $\mathbf{Cat}[a]$, etc, for $\mathbf{O}_a$, $\mathbf{E}_a$, $\mathbf{Cat}_a$.

[Problems of the notation: substitution, translation to first order, ambiguities.]

How to translate a "typical point" to first order: (..., using the diagram below):

$$\begin{array}{ll}
(b; x \Rightarrow x^F) \Rightarrow (x \overset{\bullet}{\to} ((b \to x) \to x^F)) & F_1 \\
(b; x \Rightarrow x^F) & O_1 \\
\;b & O_2 \\
\quad x \Rightarrow x^F & O_3 = F_2 \\
\quad x & O_4 \\
\qquad\quad x^F & O_5 = F_2(O_4) \\
\qquad\qquad\qquad\qquad\quad x \overset{\bullet}{\to} ((b \to x) \to x^F & O_6 = F_1(O_1) = ETN_1 = TN_1 \\
\qquad\qquad\qquad\qquad\quad x & O_7 \\
\qquad\qquad\qquad\qquad\qquad\qquad (b \to x) \to x^F & M_1 = TN_1(O_7) \\
\qquad\qquad\qquad\qquad\qquad\qquad\; b \to x & O_8 = F_3(O_7) = EM_1 = M_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\; b & O_9 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x & O_{10} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^F & O_{11} = F_4(O_7)
\end{array}$$

Some rules of the system of natural deduction for categories

$$\frac{\mathbf{O}_a}{a \to a}\,\mathrm{id} \qquad \frac{a \to b}{\mathbf{O}_a}\,\mathrm{msrc} \qquad \frac{a \to b}{\mathbf{O}_b}\,\mathrm{mtgt}$$

$$\frac{a \to b \quad b \to c}{a \to c}\,\mathrm{mm}$$

$$\frac{\mathbf{O}_a \quad a \Rightarrow a^F}{\mathbf{O}_{a^F}}\,\mathrm{OF} \qquad \frac{a \to b \quad c \Rightarrow c^F}{a^F \to b^F}\,\mathrm{mF}$$

$$\frac{\mathbf{O}_a \quad c \overset{\bullet}{\to} (c^F \to c^G)}{a^F \to a^G}\,\mathrm{OTN} \qquad \frac{a \to b \quad c \overset{\bullet}{\to} (c^F \to c^G)}{a^F \to b^G}\,\mathrm{mTN}$$

$$\frac{a \Rightarrow a^F \quad b \overset{\bullet}{\to} (b^H \to b^K)}{a \overset{\bullet}{\to} (a^{FH} \to a^{FK})}\,\mathrm{FTN} \qquad \frac{a \overset{\bullet}{\to} (a^F \to a^G) \quad b \Rightarrow b^H}{a \overset{\bullet}{\to} (a^{FH} \to b^{GH})}\,\mathrm{TNF}$$

$$\frac{\mathbf{O}_a}{\mathbf{O}_a^{\mathrm{op}}}\,\mathrm{Oop} \qquad \frac{a \to b}{b^{\mathrm{op}} \to a^{\mathrm{op}}}\,\mathrm{mop}$$

$$\frac{\mathbf{O}_a \quad \mathbf{O}_b}{\mathbf{O}_{(a;b)}}\,\mathrm{OO}\times \qquad \frac{\mathbf{O}_{(a;b)}}{\mathbf{O}_a}\,\mathrm{O}\pi_0 \qquad \frac{\mathbf{O}_{(a;b)}}{\mathbf{O}_b}\,\mathrm{O}\pi_1$$

$$\frac{a \to a' \quad b' \to b'}{(a;b) \to (a';b')}\,\mathrm{mm}\times \qquad \frac{(a;b) \to (a';b')}{a \to a'}\,\mathrm{m}\pi_0 \qquad \frac{(a;b) \to (a';b')}{b \to b'}\,\mathrm{m}\pi_1$$

$$\frac{}{(a^{\mathrm{op}};b) \Rightarrow (a \to b)}\,\mathrm{Hom}$$

$$\frac{a \to b^R}{a^L \to b}\,\flat_{LR} \qquad \frac{a^L \to b}{a \to b^R}\,\sharp_{LR}$$

$$\frac{}{a \overset{\bullet}{\to} (a \to a^{RL})}\,\eta_{LR} \qquad \frac{}{b \overset{\bullet}{\to} (b^{LR} \to b)}\,\epsilon_{LR}$$

This list is not complete.

The reason for giving a name to each rule is to let us write trees and equations in linear form; for example,

$$\dfrac{\dfrac{\mathbf{O}_a \quad \eta}{a \to a^{LR}} \quad \dfrac{\dfrac{a \to b^R}{a^L \to b} \quad b \Rightarrow b^R}{a^{LR} \to b^R}}{a \to b^R} \quad = \quad a \to b^R$$

can be written as

$$\mathrm{mm}(\mathrm{OTN}(\mathbf{O}_a, \eta_{LR}), \mathrm{mF}(\flat_{LR}(a \to b^R), (b \Rightarrow b^R))) \quad = \quad a \to b^R$$

But we will never do that in practice.

Functorial Discharges

Let $a \Rightarrow a^F$ be a functor from $\mathbf{A}$ to $\mathbf{B}$, and $b \Rightarrow b^G$ be a functor from $\mathbf{B}$ to $\mathbf{C}$; then the trees

$$\dfrac{\dfrac{\mathbf{O}[a] \quad a \Rightarrow a^F}{\mathbf{O}[a^F]} \qquad b \Rightarrow b^G}{\mathbf{O}[a^{FG}]} \qquad\qquad \dfrac{\dfrac{\mathbf{o}[\mathbf{A}] \quad \mathbf{A} \Rightarrow \mathbf{B}}{\mathbf{o}[\mathbf{B}]} \qquad \mathbf{B} \Rightarrow \mathbf{C}}{\mathbf{o}[\mathbf{C}]}$$

describe an operation that from an object $\mathbf{O}_a$ of $\mathbf{A}$ produces an object $\mathbf{O}_{a^{FG}}$ of $\mathbf{C}$; this corresponds to the the action of the functor $a \Rightarrow a^{FG}$ on objects, and syntactically the action on morphisms can be recovered from the action on objects, at least for unnamed morphisms: for example $a \Rightarrow a^{FG}$ takes $a' \to a''$ to ${a'}^{FG} \to {a''}^{FG}$; take the original morphism, erase the arrow, apply the functor on the source and target objects, draw the arrow again, and voilà, you got the image morphism, and you have used only the action on objects.

This wild idea shouldn't work in many cases, but nevertheless it suggests that if the operation represented by the tree is well-behaved enough (and this one is) it should be possible to discharge the $\mathbf{O}_a$ to form a functor, doing something like this:

$$\dfrac{\dfrac{\dfrac{[\mathbf{O}[a]]^1 \quad a \Rightarrow a^F}{\mathbf{O}[a^F]} \qquad b \Rightarrow b^G}{\mathbf{O}[a^{FG}]}}{a \Rightarrow a^{FG}}\,1 \qquad\qquad \dfrac{\dfrac{\dfrac{[\mathbf{o}[\mathbf{A}]]^1 \quad \mathbf{A} \Rightarrow \mathbf{B}}{\mathbf{o}[\mathbf{B}]} \qquad \mathbf{B} \Rightarrow \mathbf{C}}{\mathbf{o}[\mathbf{C}]}}{\mathbf{A} \Rightarrow \mathbf{C}}\,1$$

Let's replace the "generic object" $\mathbf{O}_a$ by a "generic morphism" $a' \to a''$ and adjust the rest of the tree accordingly: we get

$$\dfrac{\dfrac{a' \to a'' \quad a \Rightarrow a^F}{{a'}^F \to {a''}^F} \qquad b \Rightarrow b^G}{{a'}^{FG} \to {a''}^{FG}}$$

a tree that builds, after a series of semantically sound steps, ${a'}^{FG} \to {a''}^{FG}$ from $a' \to a''$; note that it lets us attribute a meaning to ${a'}^{FG} \to {a''}^{FG}$, as a function of $a' \to a''$.

This operation $(a' \to a'') \to ({a'}^{FG} \to {a''}^{FG})$ is what we call a *protofunctor*: it behaves as a functor $a \Rightarrow a^{FG}$ would for objects and unnamed arrows, but we haven't checked if it takes identities to identities and if it respects compositions of morphisms; technically, the bar with the discharge in

$$\dfrac{\dfrac{\dfrac{[a' \to a'']^1 \quad a \Rightarrow a^F}{{a'}^F \to {a''}^F} \qquad b \Rightarrow b^G}{{a'}^{FG} \to {a''}^{FG}}}{a \Rightarrow a^{FG}}\,1$$

is constructing only a protofunctor, but we have several reasons to pretend that it is a functor. [I will describe those reasons later, when I explain how all this fits in a method to study categories via syntactical means.]

Checking the naturality of a protonatural construction

We need to check that a protofunctor respects composition and identities:

$$\begin{array}{c} a' \to a'' \quad a'' \to a''' \quad a \Rightarrow a^F \\ \hline\hline a' \to a''' \end{array} \qquad \begin{array}{c} \mathbf{O}[a] \quad a \Rightarrow a^F \\ \hline\hline a \to a \end{array}$$

and that from an $a \leftrightarrow b$ we get $a \to b \to a = a \overset{\mathrm{id}}{\to} a$ and $b \to a \to b = b \overset{\mathrm{id}}{\to} b$,

$$\begin{array}{c} a \leftrightarrow b \\ \hline\hline a \to a \end{array} \qquad \begin{array}{c} a \leftrightarrow b \\ \hline\hline b \to b \end{array}$$

and that protonatural transformations respect the square condition:

$$\begin{array}{c} a' \to a'' \quad a \Rightarrow a^F \quad a \overset{\bullet}{\to} (a^F \to a^G) \quad a \Rightarrow a^G \\ \hline\hline {a'}^F \to {a''}^G \end{array}$$

So, for example, if we are given a protoadjunction

$$(a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a^L \to b) \leftrightarrow (a \to b^R))$$

we have to check that the two protofunctors on which it stands, $(a^{\mathrm{op}}; b) \Rightarrow (a^L \to b)$ and $(a^{\mathrm{op}}; b) \Rightarrow (a \to b^R)$, are functors, but this is a consequence of $a \Rightarrow a^L$ and $b \Rightarrow b^R$ being functors (we must check that, of course); also, we must split this NT in two and check the square condition in each one of its "directions"; this becomes so large that we'll have to use some shorthands... from

$(a^{\mathrm{op}}; b)' \to (a^{\mathrm{op}}; b)''$,
$(a^{\mathrm{op}}; b) \Rightarrow (a^L \to b)$ (derived from $a \Rightarrow a^L$),
$(a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a^L \to b) \to (a \to b^R))$, and
$(a^{\mathrm{op}}; b) \Rightarrow (a \to b^R)$ (derived from $b \Rightarrow b^L$)

there are two ways to get an $(a^L \to b)' \to (a \to b^R)''$, and they must give the same result, a tree for that is:

$$\begin{array}{c} (a^{\mathrm{op}}; b)' \to (a^{\mathrm{op}}; b)'' \quad a \Rightarrow a^L \quad (a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a^L \to b) \to (a \to b^R)) \quad b \Rightarrow b^R \\ \hline\hline (a^L \to b)' \to (a \to b^R)'' \end{array}$$

the same for taking

$(a^{\mathrm{op}}; b)' \to (a^{\mathrm{op}}; b)''$,
$(a^{\mathrm{op}}; b) \Rightarrow (a \to b^R)$ (derived from $b \Rightarrow b^L$),
$(a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a \to b^R) \to (a^L \to b))$, and
$(a^{\mathrm{op}}; b) \Rightarrow (a^L \to b)$ (derived from $a \Rightarrow a^L$)

and obtaining an $(a \to b^R)' \to (a^L \to b)''$ in two possible ways; the tree is

$$\begin{array}{c} (a^{\mathrm{op}};b)' \to (a^{\mathrm{op}};b)'' \quad b \Rightarrow b^R \quad (a^{\mathrm{op}};b) \overset{\bullet}{\to} ((a \to b^R) \to (a^L \to b)) \quad a \Rightarrow a^L \\ \hline\hline (a \to b^R)' \to (a^L \to b)'' \end{array}$$

We must also check that the two sides of the '$\leftrightarrow$' are inverses:

$$\begin{array}{c} \mathbf{O}[(a^{\mathrm{op}};b)] \quad (a^{\mathrm{op}};b) \overset{\bullet}{\to} ((a^L \to b) \leftrightarrow (a \to b^R)) \\ \hline\hline (a^L \to b) \to (a^L \to b) \end{array}$$

$$\begin{array}{c} \mathbf{O}[(a^{\mathrm{op}};b)] \quad (a^{\mathrm{op}};b) \overset{\bullet}{\to} ((a^L \to b) \leftrightarrow (a \to b^R)) \\ \hline\hline (a \to b^R) \to (a \to b^R) \end{array}$$

Checking that a diagram commutes

Let's start with the diagram below:

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[r] \arrow[d] & c \arrow[d] \\ & d \arrow[r] \arrow[dr] & e \arrow[d] \\ & & f \end{tikzcd}$$

suppose that we know the "value" of each of its six objects and eight arrows (they are our "base objects", in the sense of the last section), and we also know that its three cells commute, i.e., that $a \to b \to d = a \to d$, $b \to c \to e = b \to d \to e$ and $d \to e \to f = d \to f$. We have several natural constructions for $a \to e$, several for $b \to f$, and so on. Let's suppose that the only rules allowed in our natural constructions are composition of morphisms and generating the identity morphism for an object that is in the diagram; then in fact we have an infinite number of allowed natural constructions, as we can expand things like

$$\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{a \to a} \quad \dfrac{\mathbf{O}[a]}{a \to a}}{a \to a} \quad \dfrac{\mathbf{O}[a]}{a \to a}}{a \to a} \qquad a \to b}{a \to b}$$

to make constructions with as many levels as we want. So, how do we check that a diagram really commutes?

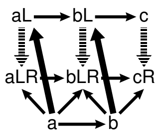
It is easy to define the "height" of a construction: base objects are constructions of height one, $\frac{a \to b \quad b \to c}{a \to c}$ has height 2, and so on. We know how to obtain all constructions of height 1 and their values, the constructions of height 2 and their values, etc; at some steps we will obtain results of new types — $b \to e$ appears first at height 2, for example — but after some height in diagrams like this new types will cease to appear, and the same with the values for each type, if we use the equivalences (and if we have enough equivalences, of course).

If we implement that with some care with could get an algorithm that examines a diagram that we know that commutes (the diagram is given together with the know equations on its arrows, like the $a \to b \to d = a \to d$ above, and the algorithm is allowed to loop forever on bad diagrams) and then it builds some "multiplication and reduction tables"; using these tables it can show, for any natural construction on that diagram, how to reduce this construction, one step at a time, using the known equalities, until reaching the first construction the algorithm has found for an object with the same value. For example

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[a]}{a \to a} \quad a \to b}{a \to b} \qquad b \to d}{a \to d} \quad \Rightarrow \quad \dfrac{a \to b \quad b \to d}{a \to d} \quad \Rightarrow \quad a \to d$$

I haven't thought very carefully about this algorithm or tried to implement it, but it seems to be close to what we do mentally when we inspect a diagram to check that it commutes — except, of course, that we are lazy and we can hold very little in our memory and we try to find a quickest way.

The diagram below is a more realistic example. It is used to prove something about adjunctions; we don't want to recall all the details now, so we'll be very sketchy. We are allowed to apply the functor L only on objects, we can choose any values for the base objects $a \to b$ and $b \to c$, and we are forbidden to refer to objects (now in the categorical sense, not in the sense of "things") that do not appear in the diagram. The arrows $a \to a^{LR}$ and $b \to b^{LR}$ can be declared base objects if we don't want to be forced to obtain them by transposing identities.



This diagram commutes, but it is slightly tricky to show that $a^L \to c$ is well-defined; we must take the two most obvious constructions for it (note that there is a transposition $(a^L \to c) \leftrightarrow (a \to c^R)$) and see that they would induce the same $a \to c^R$; so, by some '$\exists!$' argument, they must denote the same arrow.

If we want to be allowed to say "this diagram obviously commutes" in our formal language then this algorithm must be made clear; and we must do something similar for trees. At this moment we are just pretending that our formal language can hold all the "*it is obvious that*"s that are implicit in our examples.

The Yoneda Lemma

We'll quote a page or so of MacLane's CWM and show how to translate its constructions to our notation.

CWM, pages 61–62:

Lemma. *(Yoneda). If $K : D \to \mathbf{Set}$ is a functor from D and r an object in D (for D a category with small hom-sets), there is a bijection*

$$y : \mathrm{Nat}(D(r,-),K) \cong Kr$$

which sends each natural transformation $\alpha : D(r,-) \stackrel{\bullet}{\to} K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.

The proof is indicated by the following commutative diagram:

$$\begin{array}{ccccccc} & & \alpha_r & & & & \\ & D(r,r) & \to & K(r) & & r & \\ f_* = D(r,f) & \downarrow & & \downarrow & K(f) & \downarrow & f \\ & D(r,d) & \to & K(d) & & d & \\ & & \alpha_d & & & & \end{array}$$

Corollary. *For objects $r, s \in D$, each natural transformation $D(r,-) \stackrel{\bullet}{\to} D(s,-)$ has the form $D(h,-)$ for a unique arrow $h : s \to r$.*

The Yoneda map y of (4) is natural in K and r. To state this fact formally, we must consider K as an object in the functor category $\mathbf{Set}^D$, regard both domain and codomain of the map y as functors of the pair $\langle K, r\rangle$, and consider this pair as an object in the category $\mathbf{Set}^D \times D$. The codomain for y is then the evaluation functor E, which maps each pair $\langle K, r\rangle$ to the value Kr of the functor K at the object r; the domain is the functor N which maps the object $\langle K, r\rangle$ to the set $\mathrm{Nat}(D(r,-),K)$ of all natural transformations and which maps a pair of arrows $F : K \to K'$, $f : r \to r'$ to $\mathrm{Nat}(D(f,-),F)$. With these observations we may at once prove an addendum to the *Yoneda Lemma:*

Lemma. *The bijection of (4) is a natural isomorphism $y : N \stackrel{\bullet}{\to} E$ between the functors $E, N : \mathbf{Set}^D \times D \to \mathbf{Set}$.*

The object function $r \mapsto D(r,-)$ and the arrow function

$$(f : s \to r) \mapsto D(f,-) : D(r,-) \stackrel{\bullet}{\to} D(s,-)$$

for f an arrow of D together define a faithful functor

$$Y : D^{\mathrm{op}} \to \mathbf{Set}^D$$

called the *Yoneda functor*. Its dual is another such functor

$$Y' : D \to \mathbf{Set}^{D^{\mathrm{op}}}$$

(also faithful) which sends $f : s \to r$ to the natural transformation

$$D(-,f) : D(-,s) \stackrel{\bullet}{\to} D(-,r) : D^{\mathrm{op}} \to \mathbf{Set}\ .$$

D must have small hom-sets if these functors are to be defined (because **Set** is the category of all *small* sets). For larger D, the Yoneda lemmas remain valid if **Set** is replaced by any category **Ens** whose objects are sets $X, Y, \ldots$, and for which $\mathbf{Ens}(X,Y)$ is the set of all functions from X to Y, provided of course that D has hom-sets which are objects in **Ens**. (The meaning of naturality is not altered by further enlargement of **Ens**; see Exercise 4.)

The trees associated to the constructions above:

$y : \mathrm{Nat}(D(r,-),K) \to Kr$, the "forward" direction of the bijection $\mathrm{Nat}(D(r,-),K) \cong Kr$:

$$\dfrac{\dfrac{r\in D}{(1_r:r\to r)\in D(r,r)} \quad \dfrac{r\in D \quad [\alpha:D(r,-)\stackrel{\bullet}{\to}K]^1}{\alpha_r:D(r,r)\to K(r)}}{\dfrac{\alpha_r 1_r\in K(r)}{y=(\alpha\mapsto\alpha_r 1_r):\mathrm{Nat}(D(r,-),K)\to Kr}\;1}$$

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[r]}{r\to r}\quad\dfrac{\mathbf{O}[r]\quad[d\stackrel{\bullet}{\to}((r\to d)\to d^K)]^1}{(r\to r)\to r^K}}{r^K}}{(d\stackrel{\bullet}{\to}((r\to d)\to d^K))\to r^K}\;1 \qquad \dfrac{\dfrac{\dfrac{\mathbf{o}[D]}{\mathbf{m}[D],\mathbf{e}[\mathbf{E}[r\to r]]}\quad\dfrac{\mathbf{o}[D]\quad[\mathbf{m}[D\Rightarrow\mathbf{Set}]]^1}{\mathbf{m}[\mathbf{Set}]}}{\mathbf{e}[\mathbf{E}[r^K]]}}{\mathbf{e}[\mathbf{E}[(d\stackrel{\bullet}{\to}((r\to d)\to d^K))\to r^K]]}\;1$$

The other side of the bijection, $y^{-1} : Kr \to \mathrm{Nat}(D(r,-),K)$: (the last discharge step is not shown)

$$\dfrac{\dfrac{\dfrac{\alpha_r1_r\in K(r)\quad \dfrac{[(f:r\to d)\in D(r,d)]^1\quad K:D\to\mathbf{Set}}{K(f):K(r)\to K(d)}}{K(f)(\alpha_r1_r)\in K(d)}}{\alpha_d=(f\mapsto K(f)(\alpha_r1_r)):D(r,d)\to K(d)}\;1}{(\alpha:D(r,-)\stackrel{\bullet}{\to}K)\in\mathrm{Nat}(D(r,-),K)}\;2$$

$$\dfrac{\dfrac{\dfrac{r^K\quad\dfrac{[r\to d]^1\quad d\Rightarrow d^K}{r^K\to d^K}}{d^K}}{(r\to d)\to d^K}\;1}{d\stackrel{\bullet}{\to}((r\to d)\to d^K)}\;2 \qquad \dfrac{\dfrac{\dfrac{\mathbf{e}[\mathbf{E}[r^K]]\quad\dfrac{[\mathbf{m}[D]]^1\quad D\Rightarrow\mathbf{Set}}{\mathbf{m}[\mathbf{Set}]}}{\mathbf{e}[\mathbf{E}[d^K]]}}{\mathbf{m}[\mathbf{Set}]}\;1}{\mathbf{m}[D\Rightarrow\mathbf{Set}]}\;2$$

Obtaining the arrow $h : s \to r$ corresponding to a natural transformation $\alpha : D(r,-) \stackrel{\bullet}{\to} D(s,-)$, in the corollary:

$$\dfrac{\dfrac{\mathbf{O}[r]}{r\to r}\quad\dfrac{\mathbf{O}[r]\quad d\stackrel{\bullet}{\to}((r\to d)\to(s\to d))}{(r\to r)\to(s\to r)}}{s\to r}$$

$$\dfrac{\dfrac{r\in D}{(1_r:r\to r)\in D(r,r)}\quad\dfrac{r\in D\quad \alpha:D(r,-)\stackrel{\bullet}{\to}D(s,-)}{\alpha_r:D(r,r)\to D(s,r)}}{(h=\alpha_r1_r:s\to r)\in D(s,r)}$$

Newcomers generally have some difficulty with grasping the meaning of $D(h,-)$, so we'll dissect it now:

$$\dfrac{\dfrac{s\to r}{r^{\mathrm{op}}\to s^{\mathrm{op}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[r]^{\mathrm{op}}]^2 \quad [\mathbf{O}[d]]^1}{\mathbf{E}[r\to d]}}{d\Rightarrow(r\to d)}\,1}{\mathbf{O}[d\Rightarrow(r\to d)]}}{r^{\mathrm{op}}\Rightarrow(d\Rightarrow(r\to d))}\,2}{d \stackrel{\bullet}{\to} ((r\to d)\to(s\to d))} \qquad \dfrac{\dfrac{h:s\to r}{(h:s\to r)^{\mathrm{op}}} \quad \dfrac{\dfrac{\dfrac{\dfrac{[(r\in D)^{\mathrm{op}}]^2 \quad [d\in D]^1}{D(r,d)\in\mathbf{Set}}}{D(r,-):D\to\mathbf{Set}}\,1}{D(r,-)\in\mathbf{Set}^D}}{D(-,-):D^{\mathrm{op}}\to\mathbf{Set}^D}\,2}{D(h,-):D(r,-)\stackrel{\bullet}{\to}D(s,-)}$$

Now we have to check that E and N are really (proto-)functors from $\mathbf{Set}^D\times D$ to $\mathbf{Set}$.

Remember that an $\mathbf{O}_{(a;b)}$ is a pair formed by an $\mathbf{O}_a$ and an $\mathbf{O}_b$ (and object of $\mathbf{Cat}_a\times\mathbf{Cat}_b$); a morphism $(a';b')\to(a'';b'')$ is a pair $(a'\to a'';b'\to b'')$ formed by a morphism in $\mathbf{Cat}_a$ and another in $\mathbf{Cat}_b$.

We'll be writing $(r;d\Rightarrow d^K)$ for CWM's $\langle K,r\rangle$ — in our notation the functor comes last.

Here's E:

$$\dfrac{\dfrac{\dfrac{[(r';d\Rightarrow d^{K'})\to(r'';d\Rightarrow d^{K''})]^1}{r'\to r''} \quad \dfrac{\dfrac{[(r';d\Rightarrow d^{K'})\to(r'';d\Rightarrow d^{K''})]^1}{(d\Rightarrow d^{K'})\to(d\Rightarrow d^{K''})}}{d\stackrel{\bullet}{\to}(d^{K'}\to d^{K''})}\,\mathrm{ren}}{{r'}^{K'}\to{r''}^{K''}}}{(r;d\Rightarrow d^K)\Rightarrow r^K}\,1$$

$$\dfrac{\dfrac{\dfrac{[\langle F,f\rangle:\langle K,r\rangle\to\langle K',r'\rangle]^1}{f:r\to r'} \quad \dfrac{\dfrac{[\langle F,f\rangle:\langle K,r\rangle\to\langle K',r'\rangle]^1}{F:K\stackrel{\bullet}{\to}K'}}{F:K\stackrel{\bullet}{\to}K'}}{F_{r'}K(f)=K(f')F_r:Kr\to K'r'}}{E:\mathbf{Set}^D\times D\to\mathbf{Set}}\,1$$

And N (in our notation only):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[r'\to r'']^2}{{r''}^{\mathrm{op}}\to{r'}^{\mathrm{op}}} \quad \dfrac{}{r^{\mathrm{op}}\Rightarrow(d\to(r\to d))}}{d\stackrel{\bullet}{\to}((r''\to d)\to(r'\to d))} \quad [d\stackrel{\bullet}{\to}((r'\to d)\to d^{K'})]^1 \quad [d\stackrel{\bullet}{\to}(d^{K'}\to d^{K''})]^2}{d\stackrel{\bullet}{\to}((r''\to d)\to d^{K''})}}{(d\stackrel{\bullet}{\to}((r'\to d)\to d^{K'}))\to(d\stackrel{\bullet}{\to}((r''\to d)\to d^{K''}))}\,1}{(r;d\Rightarrow d^K)\Rightarrow(d\stackrel{\bullet}{\to}((r\to d)\to d^K))}\,2$$

In a well-behaved world...
(i.e., if every protonatural construction is automatically natural,)
$a \leftrightarrow b$ is just a pair $(a \to b; a \leftarrow b)$
(the same for natural transformations)
we do not need to check that protofunctors are functors
there's no square condition for NTs $a \overset{\bullet}{\to} (a^F \to a^G)$
Yoneda is just $(a; b \Rightarrow b^F) \overset{\bullet}{\to} (a^F \leftrightarrow (b \overset{\bullet}{\to} ((a \to b) \to b^F)))$
and its corollary is $(r \to s) \leftrightarrow (a \overset{\bullet}{\to} ((s \to a) \to (r \to a)))$
an adjunction is just $(a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a^L \to b) \leftrightarrow (a \to b^R))$

To show that $\mathbf{Set}^{\mathbf{C}}$ is a topos we must prove that it has finite limits (which I am not doing in these notes) and:
$\mathbf{O}[x] := c \Rightarrow x_c$, $\mathbf{O}[y] := c \Rightarrow y_c$, etc

$\mathbf{O}[(x \to y)] := c \Rightarrow ((c \to \underline{d}) \overset{\bullet}{\to} (x_d \to y_d))$
$y^{\mathrm{op}} \Rightarrow ((x^{\mathrm{op}}, z) \overset{\bullet}{\to} ((x, y \to z) \leftrightarrow (x \to (y \to z))))$

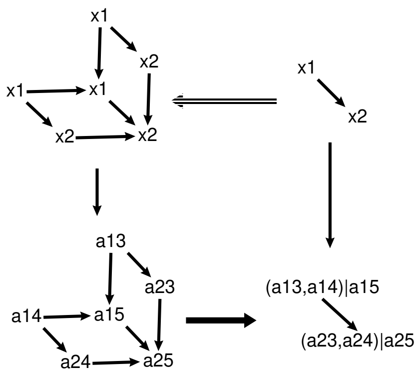
$\mathbf{O}_\omega := c \Rightarrow \mathrm{sub}((c \to \underline{d}) \Rightarrow \top)$
$x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to \omega) \leftrightarrow \mathrm{sub}(x))$

Just a comment about limits in $\mathbf{Set}^{\mathbf{C}}$, as we won't prove formally that they exist (the sketch I have for a syntactical proof of this is not convincing).

A $\mathbf{Set}^{\mathbf{C}}$ inherits limits and colimits from $\mathbf{Set}$, in that they can be computed "coordinatewise in $\mathbf{C}$"; in $\mathbf{Set}$ limits appear from the adjunction
$(x^{\mathrm{op}}; i \Rightarrow a_i) \overset{\bullet}{\to} (((i \Rightarrow x) \to (i \Rightarrow a_i)) \leftrightarrow (x \to \lim(i \Rightarrow a_i)))$
$(x^{\mathrm{op}}; i \Rightarrow a_i) \overset{\bullet}{\to} ((i \overset{\bullet}{\to} (x \to a_i)) \leftrightarrow (x \to \lim(i \Rightarrow a_i)))$

$$\begin{array}{ccc}
(i \Rightarrow x) & \Leftarrow & x \\
\downarrow & \leftrightarrow & \downarrow \\
(i \Rightarrow a_i) & \Rightarrow & \lim(i \Rightarrow a_i)
\end{array}$$

To transport that to a $\mathbf{Set}^{\mathbf{C}}$ we must replace x by $c \Rightarrow x_c$ and a_i by $c \Rightarrow a_{ci}$; the trick is that it is possible to prove that we can use $c \Rightarrow \lim(i \Rightarrow a_{ci})$ in the adjunction, instead of the more complex $\lim(i \Rightarrow (c \Rightarrow a_{ci}))$; in fact we can define $\lim(i \Rightarrow (c \Rightarrow a_{ci})) := c \Rightarrow \lim(i \Rightarrow a_{ci})$...



Universal Elements

Sometimes when we plug a functor $b \Rightarrow b^F$, an object $\mathbf{O}_a$ and a point of $\mathbf{E}_{a^F}$ into the Yoneda lemma,

$$\dfrac{a^F \qquad \dfrac{\mathbf{O}[a] \qquad \dfrac{b \Rightarrow b^F \quad (a; b \Rightarrow b^F) \overset{\bullet}{\to} (a^F \leftrightarrow (b \overset{\bullet}{\to} ((a \to b) \to b^F)))}{a \overset{\bullet}{\to} (a^F \leftrightarrow (b \overset{\bullet}{\to} ((a \to b) \to b^F)))}}{a^F \leftrightarrow (b \overset{\bullet}{\to} ((a \to b) \to b^F))}}{b \overset{\bullet}{\to} ((a \to b) \to b^F)}$$

the natural transformation that we get is a natural isomorphism, i.e., it can be written as $b \overset{\bullet}{\to} ((a \to b) \leftrightarrow b^F)$; when that happens we say that a^F is a *universal element* for $b \Rightarrow b^F$, and that $b \to b^F$ is a *representable functor*, represented by $\mathbf{O}_a$. Actually our notation lets us infer $b \Rightarrow b^F$ and $\mathbf{O}_a$ from an $a^F \leftrightarrow (b \overset{\bullet}{\to} ((a \to b) \to b^F))$, and the a^F corresponding to a $b \overset{\bullet}{\to} ((a \to b) \to b^F)$ can be obtained by

$$\dfrac{\dfrac{\mathbf{O}[a]}{a \to a}\,\mathrm{id} \qquad \dfrac{\mathbf{O}[a] \quad b \overset{\bullet}{\to} ((a \to b) \to b^F)}{(a \to a) \to a^F}}{a^F}$$

so we generally speak of representable functors and universal elements for them by showing the "representation of the functor", the natural isomorphism $b \overset{\bullet}{\to} ((a \to b) \to b^F)$; this may be a bit longer sometimes, but is much more expressive.

A first example: let $O_{c \amalg d}$ in $\mathbf{C}$ be a coproduct of $\mathbf{O}_c$ and $\mathbf{O}_d$ (in **Set** $\mathbf{E}_{c \amalg d}$ is the disjoint union of $\mathbf{E}_c$ and $\mathbf{E}_d$), with natural inclusions $c \to c \amalg d$ and $d \to c \amalg d$; let $x \Rightarrow (c \to x; d \to x)$ be the functor from $\mathbf{C}$ to **Set** that carries each object $\mathbf{O}_x$ to the set of pairs of arrows with the right sources $\mathbf{E}_{(c \to x; d \to x)}$. We have

$$x \overset{\bullet}{\to} ((c \amalg d \to x) \leftrightarrow (c \to x; d \to x))$$

associated to the universal element $(c \to c \amalg d; d \to c \amalg d)$.

We also have something similar for contravariant functors; in

$$\dfrac{a^F \qquad \dfrac{\mathbf{O}[a]^{\mathrm{op}} \qquad \dfrac{b^{\mathrm{op}} \Rightarrow b^F \quad (a^{\mathrm{op}}; b^{\mathrm{op}} \Rightarrow b^F) \overset{\bullet}{\to} (a^F \leftrightarrow (b^{\mathrm{op}} \overset{\bullet}{\to} ((b \to a) \to b^F)))}{a^{\mathrm{op}} \overset{\bullet}{\to} (a^F \leftrightarrow (b^{\mathrm{op}} \overset{\bullet}{\to} ((b \to a) \to b^F)))}}{a^F \leftrightarrow (b^{\mathrm{op}} \overset{\bullet}{\to} ((b \to a) \to b^F))}}{b^{\mathrm{op}} \overset{\bullet}{\to} ((b \to a) \to b^F)}$$

when we get a $b^{\mathrm{op}} \overset{\bullet}{\to} ((b \to a) \leftrightarrow b^F)$ we say that the contravariant functor $b^{\mathrm{op}} \Rightarrow b$ is represented by $\mathbf{O}_a$, with couniversal element a^F.

As an example of this, fix arrows $a \to c$ and $b \to c$, and take $x^{\mathrm{op}} \Rightarrow (x \to a; x \to b)|_c$, the functor that carries each $\mathbf{O}_x$ to the set of pairs of arrows with targets $\mathbf{O}_a$ and $\mathbf{O}_b$ such that "$x \to c$" is well-defined, i.e., such that $x \to a \to c = x \to b \to c$. This is represented by

$$x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to (a,b)|_c) \leftrightarrow (x \to a; x \to b)|_c),$$

and so pullbacks fit in this form.

$x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to \lim(i \Rightarrow a_i)) \leftrightarrow ((i \Rightarrow x) \to (i \Rightarrow a_i)))$
or
$x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to \lim(i \Rightarrow a_i)) \leftrightarrow (i \overset{\bullet}{\to} (x \to a_i)))$
And as an adjunction:
$(x^{\mathrm{op}}; i \Rightarrow a_i) \overset{\bullet}{\to} (((i \Rightarrow x) \to (i \to a_i)) \leftrightarrow (x \to \lim(i \Rightarrow a_i)))$
$(x^{\mathrm{op}}; i \Rightarrow a_i) \overset{\bullet}{\to} ((x \to \lim(i \Rightarrow a_i)) \leftrightarrow (i \overset{\bullet}{\to} (x \to a_i)))$
$(x \Rightarrow (i \Rightarrow x)) \dashv ((i \Rightarrow a_i) \Rightarrow \lim(i \Rightarrow a_i))$
$((i \Rightarrow a_i) \Rightarrow \mathrm{colim}(i \Rightarrow a_i)) \dashv (x \Rightarrow (i \Rightarrow x))$

Categories of morphisms

(Note: the use of underlines here is slightly different from what I've been using in older notes.)

An $\mathbf{O}_{\underline{a}\to\underline{b}}$, $\mathbf{O}_{\underline{a}\to b}$ or $\mathbf{O}_{a\to\underline{b}}$ is a morphism $a \to b$.

An $(\underline{a'} \to \underline{b'}) \to (\underline{a''} \to \underline{b''})$ is a pair $(a' \to a''; b' \to b'')$ such that the obvious diagram commutes;

an $(\underline{a'} \to b) \to (\underline{a''} \to b)$ is an $a' \to a''$ such that the diagram commutes;

an $(a \to \underline{b'}) \to (a \to \underline{b''})$ is a $b' \to b''$ such that the diagram commutes.

The underlines indicate where we have non-trivial projection functors: we have $(\underline{a} \to \underline{b}) \Rightarrow a$, $(\underline{a} \to \underline{b}) \Rightarrow b$, $(\underline{a} \to b) \Rightarrow a$, $(a \to \underline{b}) \Rightarrow b$. All '$\mathbf{O}_{\underline{a}\to b}$'s have the same target object $\mathbf{O}_b$, and all '$\mathbf{O}_{a\to\underline{b}}$'s have the same source object $\mathbf{O}_a$, so projecting on a non-underlined coordinate would be trivial.

We have inclusion functors $(\underline{a} \to b) \Rightarrow (\underline{a} \to \underline{b})$ and $(a \to \underline{b}) \Rightarrow (\underline{a} \to \underline{b})$.

A morphism $a' \to a''$ induces a functor $(a'' \to \underline{b}) \Rightarrow (a' \to \underline{b})$.

A morphism $b' \to b''$ induces a functor $(\underline{a} \to b') \Rightarrow (\underline{a} \to b'')$.

Let's use a notational trick: if we downcase $\mathbf{Cat}[(\underline{a} \to b')]$ to $\mathbf{cat}[(\underline{a} \to b')]$ we will be regarding that category as an object of a category of categories. A functor $(\underline{a} \to b') \Rightarrow (\underline{a} \to b'')$ is a morphism from $\mathbf{cat}[(\underline{a} \to b')]$ to $\mathbf{cat}[(\underline{a} \to b'')]$, and so we have

$$\cfrac{\cfrac{\cfrac{[b' \to b'']^1}{(\underline{a} \to b') \Rightarrow (\underline{a} \to b'')}}{\mathbf{cat}[(\underline{a} \to b')] \to \mathbf{cat}[(\underline{a} \to b'')]}}{b \Rightarrow \mathbf{cat}[(\underline{a} \to b)]}\;1$$

and we get a functor $b \Rightarrow \mathbf{cat}[(\underline{a} \to b)]$; similarly,

$$\cfrac{\cfrac{\cfrac{[a' \to a'']^1}{(a' \to \underline{b}) \Leftarrow (a'' \to \underline{b})}}{\mathbf{cat}[(a' \to \underline{b})] \leftarrow \mathbf{cat}[(a'' \to \underline{b})]}}{a^{\mathrm{op}} \Rightarrow \mathbf{cat}[(a \to \underline{b})]}\;1 \qquad .$$

We will use double underlines to switch the direction of component arrows: we have projections $(\underline{\underline{a}} \to \underline{b}) \Rightarrow a^{\mathrm{op}}$ and $(\underline{\underline{a}} \to \underline{b}) \Rightarrow b$, for example. Note that by composing $(\underline{\underline{a}} \to \underline{b}) \Rightarrow (a^{\mathrm{op}}; b)$ with Hom (i.e., with $(a^{\mathrm{op}}; b) \Rightarrow (a \to b)$) we may regard each $\mathbf{O}_{(\underline{\underline{a}}\to\underline{b})}$ as corresponding to a specific point of $\mathbf{E}_{a\to b}$, and each morphism $(\underline{\underline{a'}} \to \underline{b'}) \to (\underline{\underline{a''}} \to \underline{b''})$ as a morphism $(a' \to b') \to (a'' \to b'')$ that takes the specified point in the first space of morphisms to the specified point in the second space of morphisms.

Now take a functor $c \Rightarrow c^F$ from $\mathbf{C}$ to $\mathbf{A}$, and a functor $d \Rightarrow d^G$ from $\mathbf{D}$ to $\mathbf{A}$. An object $\mathbf{O}[(\underline{c}^F \to \underline{d}^G)]$ is a triple $(\mathbf{O}_c, c^F \to d^G, \mathbf{O}_d)$; a morphism $(\underline{c'}^F \to \underline{d'}^G) \to (\underline{c''}^F \to \underline{d''}^G)$ is a pair of morphisms $(c' \to c''; d' \to d'')$, one in $\mathbf{C}$ and the other in $\mathbf{D}$, such that the obvious diagram in $\mathbf{A}$ commutes. We have projections $(\underline{c}^F \to \underline{d}^G) \Rightarrow c$ and $(\underline{c}^F \to \underline{d}^G) \Rightarrow d$.

The usual notation for $\mathbf{Cat}[(\underline{c}^F \to \underline{d}^G)]$ is (F, G), and this is called the **comma category** (F, G) (see Barr/Wells, pp.13–14, or CWM, p.47).

If $\mathbf{O}_a, \mathbf{O}_{a'}, \ldots, \mathbf{O}_b, \mathbf{O}_{b'}, \ldots$ are objects of $\mathbf{C}$, then $\mathbf{Cat}[(\underline{a} \to \underline{b})]$ is the **category of arrows** of $\mathbf{C}$ (called $\mathrm{Ar}(\mathcal{C})$ in Barr/Wells p.9, $\mathbf{C^2}$ in CWM p.40). In the same page of Barr/Wells the **twisted category of arrows of** $\mathcal{C}$ is defined; it is equivalent to our $\mathbf{Cat}[(\underline{a} \to \underline{\underline{b}})]$.

Slice categories: Barr and $\overline{\text{W}}$ells' $\mathbf{C}/A$ (p.3) is our $\mathbf{Cat}[(\underline{b} \to a)]$; CWM (p.46) calls this the **category of objects over** a, $(C \downarrow a)$ (our $\mathbf{C}$ is CWM's C; our $\mathbf{O}_a$ is CWM's a).

Slice, comma and pre-comma categories:

Barr and Wells, pp.13–14:

Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{A}$, $G : \mathcal{D} \to \mathcal{A}$ be functors. From these ingredients we construct the **comma category** (F, G) which is a generalization of the slice $\mathcal{A}/A$ of a category over an object discussed in Section 1.1. The objects of (F, G) are triples (C, f, D) with $f : FC \to FD$ an arrow of $\mathcal{A}$ and C, D objects of $\mathcal{C}$ and $\mathcal{D}$ respectively. An arrow $(h, k) : (C, f, D) \to (C', f', D')$ consists of $h : C \to C'$ and $k : D \to D'$ making

$$\begin{array}{ccccc} & & Fh & & \\ & FC & \to & FC' & \\ f & \downarrow & & \downarrow & f' \\ & GD & \to & GD' & \\ & & Gk & & \end{array}$$

commute. It is easy to verify that coordinatewise composition makes (F, G) a category.

When A is an object of $\mathcal{A}$, we can consider it as a functor $A : 1 \to \mathcal{A}$. Then the comma category $(\mathbf{Id}_{\mathcal{A}}, A)$ is just the slice $\mathcal{A}/A$ defined in Section 1.1. The category of arrows under an object is similarly a comma category.

Each comma category (F, G) is equipped with two **projections** $p_1 : (F, G) \to \mathcal{C}$ projecting objects and arrows onto their first coordinates, and $p_2 : (F, G) \to \mathcal{D}$ projecting objects onto their thrid coordinates and arrows onto their second.

The same in my notation:

Let $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$ be categories and $c \Rightarrow c^F : \mathbf{C} \Rightarrow \mathbf{A}$, $d \Rightarrow d^G : \mathbf{D} \to \mathbf{A}$ be functors. From these ingredients we construct the **comma category** $\mathbf{Cat}[(\underline{c}^F \to \underline{d}^G)]$ which is a generalization of the slice $\mathcal{A}/A$ of a category over an object discussed in Section 1.1. The objects of $\mathbf{Cat}[(\underline{c}^F \to \underline{d}^G)]$ are triples $\mathbf{O}_{(\underline{c}^F \to \underline{d}^G)} := (\mathbf{O}_c, c^F \to d^G, \mathbf{O}_d)$ with $c^F \to d^G$ an arrow of $\mathbf{A}$ and $\mathbf{O}_c$, $\mathbf{O}_d$ objects of $\mathbf{C}$ and $\mathbf{D}$ respectively. An arrow $(\underline{c}^F \to \underline{d}^G) \to (\underline{c'}^F \to \underline{d'}^G)$ consists of $c \to c'$ and $d \to d'$ making [the middle square in]

$$\begin{array}{ccc} c & \to & c' \\ c^F & \to & {c'}^F \\ \downarrow & & \downarrow \\ d^G & \to & {d'}^G \\ d & \to & d' \end{array}$$

commute. It is easy to verify that coordinatewise composition makes $\mathbf{Cat}[(\underline{c}^F \to \underline{d}^G)]$ a category.

When $\mathbf{O}_a$ is an object of $\mathbf{A}$, we can consider it as a functor $A : 1 \to \mathcal{A}$. Then the comma category $(\mathbf{Id}_{\mathcal{A}}, A)$ is just the slice $\mathcal{A}/A$ defined in Section 1.1. The category of arrows under an object is similarly a comma category.

Each comma category (F, G) is equipped with two **projections** $p_1 : (F, G) \to \mathcal{C}$ projecting objects and arrows onto their first coordinates, and $p_2 : (F, G) \to \mathcal{D}$ projecting objects onto their third coordinates and arrows onto their second.

Exponentials in the Sierpinski topos

Argumento pra encontrar as exponenciais no "espaço de Sierpinski" $\mathbf{Set}^{a\to b}$, onde esse $a \to b$ é uma categoria com dois objetos e um morfismo não-trivial:

$$\begin{array}{ccc} x,y & \Leftarrow & x \\ \downarrow & \leftrightarrow & \downarrow \\ z & \Rightarrow & y \to z \end{array}$$

nessa categoria vira

$$\begin{array}{ccccccc} x_a, y_a & \to & x_b, y_b & \Leftarrow & x_a & \to & x_b \\ & \downarrow & & \leftrightarrow & & \downarrow & \\ z_a & \to & z_b & \Rightarrow & (y \to z)_a & \to & (y \to z)_b \end{array}$$

que na verdade é issso aqui, porque as setas verticais são transformações naturais:

$$\begin{array}{ccccccc} x_a, y_a & \to & x_b, y_b & \Leftarrow & x_a & \to & x_b \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ z_a & \to & z_b & \Rightarrow & (y \to z)_a & \to & (y \to z)_b \end{array}$$

No caso em que $\mathbf{E}[x_a] = \mathbf{E}_\bot = \varnothing$ e $\mathbf{E}[x_b] = \mathbf{E}_\top = \{\top\}$ isso vira

$$\begin{array}{ccccccc} \bot & \to & y_b & \Leftarrow & \bot & \to & \top \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ z_a & \to & z_b & \Rightarrow & (y \to z)_a & \to & (y \to z)_b \end{array}$$

e a gente deduz que $\mathbf{E}[(y \to z)_b] = \mathbf{E}[y_b \to z_b]$; se $\mathbf{E}[x_a] = \mathbf{E}[x_b] = \mathbf{E}_\top = \{\top\}$ isso vira

$$\begin{array}{ccccccc} y_a & \to & y_b & \Leftarrow & \top & \to & \top \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ z_a & \to & z_b & \Rightarrow & (y \to z)_a & \to & y_b \to z_b \end{array}$$

e a preimagem de cada $y_b \to z_b$ por $(y \to z)_a \to (y_b \to z_b)$ tem que ser o conjunto dos $y_a \to z_a$ que fazem um certo diagrama (incidentalmente o do quadrado à esquerda na figura acima) comutar; ou seja, temos

$$\begin{array}{ccccccc} y_a & \to & y_b & \Leftarrow & \top & \to & \top \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ z_a & \to & z_b & \Rightarrow & (y_a \to z_a; y_b \to z_b)|_{\mathrm{cmt}} & \to & (y \to z)_b \end{array}$$

[Repare que a "informação vai diminuindo" quando a gente vai para um mundo mais adiante no $y \to z$; o motivo dessa (huh) assimetria é que vários pontos à esquerda podem ir no mesmo ponto à direita, mas não o contrário; se eu tenho um ponto de $(y \to z)_a$ eu automaticamente tenho o $(y \to z)_b$.]

Exponentials and the classifier in a $\mathbf{Set}^{\mathbf{C}}$, syntactically

$$\dfrac{\mathbf{O}[x]\quad\mathbf{O}[y]}{\mathbf{O}[x,y]}\qquad\dfrac{\mathbf{O}[y]^{\mathrm{op}}\quad\mathbf{O}[z]}{\mathbf{O}[y\to z]}$$

$$\dfrac{x,y\to z}{x\to(y\to z)}\qquad\dfrac{x\to(y\to z)}{x,y\to z}$$

$$\dfrac{}{x,(x\to y)\to y}\qquad\dfrac{}{x\to(y\to x,y)}$$

$$\dfrac{\dfrac{\dfrac{[c'\to c'']^1}{\omega_{c'}\to\omega_{c''}}}{c\Rightarrow\omega_c}\,1}{\mathbf{O}[\omega]}\,\mathrm{ren}\qquad\dfrac{}{\top\to\omega}\qquad\dfrac{x\to\omega\quad\dfrac{}{\top\to\omega}}{x'\hookrightarrow x}\,\mathrm{pb}$$

$$\dfrac{x'\hookrightarrow x}{x\to\omega}$$

Here's how to build $\mathbf{O}_{(y\to z)_c} = \mathbf{E}_{(y\to z)_c}$ for a given object $\mathbf{O}_c$, and at the same time show that this construction is contravariant on $\mathbf{O}_y$ and covariant on $\mathbf{O}_z$; Note that to discharge $\mathbf{O}_{(c\to\underline{d})}$ functorially we need to know the category where this object lives, and to build it we need not only $\mathbf{Cat}_c = \mathbf{Cat}_d$ but also $\mathbf{O}_c$ — thus the tree below has a hidden dependency on $\mathbf{O}_c$, which explains why it builds only $(y^{\mathrm{op}}; z) \Rightarrow (y \to z)_c$ and not $(y^{\mathrm{op}}; z) \Rightarrow (c \Rightarrow (y \to z))$, which would be $(y^{\mathrm{op}}; z) \Rightarrow (y \to z)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[c\Rightarrow y_c]^{\mathrm{op}}]^4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[(c\to\underline{d})]]^1}{\mathbf{O}[d]} \quad [c\Rightarrow y_c]^2}{\mathbf{O}[y_d]}}{(c\to\underline{d})\Rightarrow y_d}\;1}{\mathbf{O}[(c\to\underline{d})\Rightarrow y_d]}}{\mathbf{O}[(c\to\underline{d})\Rightarrow y_d]^{\mathrm{op}}}\;2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[(c\to\underline{d})]]^3}{\mathbf{O}[d]} \quad [c\Rightarrow z_c]^4}{\mathbf{O}[z_d]}}{(c\to\underline{d})\Rightarrow z_d}\;3}{\mathbf{O}[(c\to\underline{d})\Rightarrow z_d]}}{\mathbf{E}[((c\to\underline{d})\Rightarrow y_d)\to((c\to\underline{d})\Rightarrow z_d)]}}{\mathbf{E}[(c\to\underline{d})\overset{\bullet}{\to}(y_d\to z_d)]}\;\mathrm{ren}}{\dfrac{((c\Rightarrow y_c)^{\mathrm{op}}; c\Rightarrow z_c)\Rightarrow((c\to\underline{d})\overset{\bullet}{\to}(y_d\to z_d))}{(y^{\mathrm{op}}; z)\Rightarrow(y\to z)_c}\;\mathrm{ren}}\;4
$$

This is functorial on $\mathbf{O}_c$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[c'\to c'']^3 \quad \dfrac{\mathbf{O}[(c''\to\underline{d})]^1}{c''\to d}}{c'\to d}}{\mathbf{O}[(c'\to\underline{d})]} \qquad [(c'\to\underline{d})\overset{\bullet}{\to}(y_d\to z_d)]^2}{y_d\to z_d}}{(c''\to\underline{d})\to(y_d\to z_d)}\;1}{((c'\to\underline{d})\overset{\bullet}{\to}(y_d\to z_d))\to((c''\to\underline{d})\to(y_d\to z_d))}\;2}{c\Rightarrow((c\to\underline{d})\overset{\bullet}{\to}(y_d\to z_d))}\;3}{c\Rightarrow(y\to z)_c}\;\mathrm{ren}}{\mathbf{O}[y\to z]}\;\mathrm{ren}
$$

Uncurrying is easy:

$$
\dfrac{\dfrac{[y_c]^1 \qquad \dfrac{\dfrac{\dfrac{[\mathbf{O}[c]]^2}{c \to c}}{\mathbf{O}[c \to c]} \qquad \dfrac{[x_c]^1 \qquad \dfrac{[\mathbf{O}[c]]^2 \qquad \dfrac{x \to (y \to z)}{c \overset{\bullet}{\to} (x_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d)))}\,\text{ren}}{x_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d))}}{(c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d)}}{y_c \to z_c}}{\dfrac{z_c}{\overline{\overline{x_c, y_c \to z_c}}}\;1}}{\dfrac{\dfrac{}{c \overset{\bullet}{\to} (x_c, y_c \to z_c)}\;2}{x, y \to z}\;\text{ren}}
$$

Currying is trickier, as at first sight $x \to (y \to z)$ “carries more information” than $x, y \to z$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x_c]^2 \qquad \dfrac{\dfrac{[\mathbf{O}[(c \to \underline{d})]]^1}{c \to d} \qquad \dfrac{\mathbf{O}[x]}{a \Rightarrow x_a}}{x_c \to x_d}}{x_d} \qquad \dfrac{\dfrac{[\mathbf{O}[(c \to \underline{d})]]^1}{\mathbf{O}[d]} \qquad \dfrac{x, y \to z}{c \overset{\bullet}{\to} (x_c, y_c \to z_c)}\,\text{ren}}{x_d, y_d \to z_d}}{\overline{\overline{y_d \to z_d}}}}{(c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d)}\;1}{x_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d))}\;2}{c \overset{\bullet}{\to} (x_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (y_d \to z_d)))}\;3}{x \to (y \to z)}\;\text{ren}
$$

The counit (?) of the adjunction, 'ev': $(x,(x \to y)) \to y$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x_c]^1 \qquad \dfrac{\dfrac{\dfrac{[\mathbf{O}[c]]^2}{c \to c}}{\mathbf{O}[(c \to \underline{c})]} \quad [(c \to \underline{d}) \overset{\bullet}{\to} (x_d \to y_d)]^1}{x_c \to y_c}}{y_c}}{x_c, ((c \to \underline{d}) \overset{\bullet}{\to} (x_d \to y_d)) \to y_c}\;2}{c \overset{\bullet}{\to} (x_c, ((c \to \underline{d}) \overset{\bullet}{\to} (x_d \to y_d)) \to y_c)}\;\text{ren}}{x, (x - y) \to y}
$$

The unit (?), $(x,(x \to y)) \to y$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[y_c]^2 \qquad \dfrac{\dfrac{\mathbf{O}[(c \to \underline{d})]^1}{c \to d} \quad c \Rightarrow y_c}{y_c \to y_d}}{y_d}}{x_d \to x_d, y_d}\;1}{(c \to \underline{d}) \overset{\bullet}{\to} (x_d \to x_d, y_d)}\;2}{y_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (x_d \to x_d, y_d))}\;3}{c \overset{\bullet}{\to} (y_c \to ((c \to \underline{d}) \overset{\bullet}{\to} (x_d \to x_d, y_d)))}\;\text{ren}}{y \to (x \to x, y)}
$$

Building the classifier object and checking that it is a functor:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[c'\to c'']^3 \quad \dfrac{[\mathbf{O}[(c''\to\underline{d})]]^1}{c''\to d}}{c'\to d}}{\mathbf{O}[(c'\to\underline{d})]} \qquad \dfrac{[\mathrm{sub}((c'\to\underline{d})\Rightarrow\top)]^2}{(c'\to\underline{d})\Rightarrow(\top|_S)_{(c'\to\underline{d})}}\;\mathrm{ren}}{\mathbf{O}[(\top|_S)_{(c'\to\underline{d})}]}}{(c''\to\underline{d})\Rightarrow(\top|_S)_{(c'\to\underline{d})}}\;1}{(c''\to\underline{d})\Rightarrow(\top|_S)_{(c''\to\underline{d})}}\;\mathrm{ren}}{\mathrm{sub}((c''\to\underline{d})\Rightarrow\top)}\;\mathrm{ren}}{\mathrm{sub}((c'\to\underline{d})\Rightarrow\top)\to\mathrm{sub}((c''\to\underline{d})\Rightarrow\top)}\;2}{c\Rightarrow\mathrm{sub}((c\to\underline{d})\Rightarrow\top)}\;3}{c\Rightarrow\omega_c}\;\mathrm{ren}}{\mathbf{O}[\omega]}\;\mathrm{ren}
$$

The ‘true’ arrow:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{E}[\top]}{(c\to\underline{d})\Rightarrow\top}\;1}{\mathrm{sub}((c\to\underline{d})\Rightarrow\top)}}{\top\hookrightarrow\mathrm{sub}((c\to\underline{d})\Rightarrow\top)}\;2}{c\stackrel{\bullet}{\to}(\top\hookrightarrow\mathrm{sub}((c\to\underline{d})\Rightarrow\top))}\;3}{\top\hookrightarrow\omega}\;\mathrm{ren}
$$

The classifier rule (that given a monic obtains its characteristic function):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[(c\to\underline{d})]]^1}{\mathbf{O}[d]} \quad \dfrac{x'\hookrightarrow x}{c\overset{\bullet}{\to}(x'_c\hookrightarrow x_c)}\,\text{ren}}{x'_d\hookrightarrow x_d} \quad \dfrac{[x_c]^2 \quad \dfrac{\dfrac{[\mathbf{O}[(c\to\underline{d})]]^1}{c\to d} \quad \dfrac{\dfrac{x'\hookrightarrow x}{\mathbf{O}[x]}}{c\Rightarrow x_c}\,\text{ren}}{x_c\to x_d}}{x_d}}{\top|_{\exists x'_d}}}{(c\to\underline{d})\Rightarrow\top|_{\exists x'_d}}\,1}{\text{sub}((c\to\underline{d})\Rightarrow\top)}\,\text{ren}}{x_c\to\text{sub}((c\to\underline{d})\Rightarrow\top)}\,2}{c\overset{\bullet}{\to}(x_c\to\text{sub}((c\to\underline{d})\Rightarrow\top))}\,3}{x\to\omega}\,\text{ren}
$$

Pullback with the 'true' arrow:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[c]]^2}{c\to c}}{\mathbf{O}[c\to c]} \quad \dfrac{\dfrac{[x_c]^1 \quad \dfrac{[\mathbf{O}[c]]^2 \quad \dfrac{x\to\omega_S}{c\overset{\bullet}{\to}(x_c\to\text{sub}((c\to\underline{d})\Rightarrow\top))}\,\text{ren}}{x_c\to\text{sub}((c\to\underline{d})\Rightarrow\top)}}{\text{sub}((c\to\underline{d})\Rightarrow\top)}}{(c\to\underline{d})\overset{\bullet}{\to}((\top|_S)_{(c\to\underline{d})}\hookrightarrow\top)}}{(\top|_S)_{(c\to\underline{c})}\hookrightarrow\top}}{\top\to\omega_S}}{\omega_S}}{x_c\to\omega_S}\,1}{(x|_S)_c\hookrightarrow x_c}}{c\overset{\bullet}{\to}((x|_S)_c\hookrightarrow x_c)}\,2}{x|_S\hookrightarrow x}\,\text{ren}
$$