Sheaves Over Finite DAGs May be Archetypal

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We hear that Category Theory is very useful for Logic, that Topos Theory is central in Category Theory, and that toposes came from the observation that sheaves "behave like sets". However, look at any book on Topos Theory and you will find that:

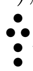
(1) the precise definition of "sheaf" is quite involved,

(2) a sheaf is a presheaf that obeys two conditions: "separatedness" and "collatedness". Presheaves are easy to define, but examples of presheaves that fail these two conditions are very artificial to say the least,

(3) there are *two* different definitions of sheaf: sheaves on topological spaces and sheaves on sites, and sheaves on sites are more general,

(4) each closure operation induces a sheaf, and $P \mapsto \neg\neg P$, $P \mapsto P \vee \alpha$, $P \mapsto (\beta \to P)$ are closure operations, where α and β are (intuitionistic) truth-values.

In this talk we will present a way to make all these ideas concrete. Take any subset $D \subseteq \mathbb{Z}^2$ and build the set of "black pawn's moves" on it, D_R, and its reflexive-transitive closure, D_R^*. Then (D, D_R) is a DAG, and the poset (D, D_R^*) can be regarded as a category ("$\mathbf{D}$") and as modal frame for S4; the category $\mathbf{Set}^{\mathbf{D}}$ is a topos of presheaves, and its truth-values are exactly the truth-values P in that modal frame obeying $P = \Box P$.

Consider for example $V \equiv$ . The truth-values of $\mathbf{Set}^{\mathbf{V}}$ can be represented as ${}^{0\,0}_{\ 0}$, ${}^{0\,0}_{\ 1}$, ${}^{0\,1}_{\ 1}$, ${}^{1\,0}_{\ 1}$, ${}^{1\,1}_{\ 1}$. These truth-values are exactly the open sets of $(V, \mathcal{O}(V))$, where $\mathcal{O}(V)$ is the order topology induced by V_R^*, and it turns out that whenever we start with a subset $D \subset \mathbb{Z}^2$ that is "thin enough", the derived DAG $D' := (\mathcal{O}(D), \supsetneq)$ can also be represented as a subset of $\mathbb{Z}^2$, which is also "thin"; for example, ()$' \equiv$ .

A *thin ZDAG* is a DAG induced by a "thin" finite subset of $\mathbb{Z}^2$. For any thin ZDAG D all the mathematical objects involved in the definition of sheaves on $\mathbf{Set}^{\mathbf{D}}$, $\mathbf{Set}^{\mathbf{D}'}$, etc — and this includes both "topological sheaves" and "site sheaves" — can be represented as planar diagrams, and can be drawn explicitly. I have the impression that sheaves on thin ZDAGs are "archetypal", in the sense of [1], but there are still some details to be verified. *This is a work in progress.*

References

[1] E. Ochs. Internal Diagrams in Category Theory. Submitted (to *Logica Universalis*).

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